Combinatorial Solving with Provably Correct Results

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KU Leuven & Vrije Universiteit Brussel

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DTAI
Declarative Languages & Artificial Intelligence

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ARTIFICIAL INTELLIGENCE RESEARCH GROUP

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ABOUT THIS TALK

- Based on (small port of) tutorial co-developed with Ciaran McCreesh and Jakob Nordström
- Useful material: `https://www.bartbogaerts.eu/talks/veripb-tutorial-series`

OUTLINE

1. Introduction
   1. The Success of Combinatorial Solving (and the Dirty Little Secret...)
   2. Ensuring Correctness with the Help of Proof Logging
   3. This Seminar
2. Proof Logging for SAT
   1. SAT Basics
   2. DPLL and CDCL
   3. Proof System for SAT Proof Logging
3. Pseudo-Boolean Proof Logging
   1. Pseudo-Boolean Constraints and Cutting Planes Reasoning
   2. Pseudo-Boolean Proof Logging for SAT Solving
   3. More Pseudo-Boolean Proof Logging Rules
4. Conclusions
   1. Future Work
   2. Concluding Remarks

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COMBINATORIAL SOLVING AND OPTIMISATION

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong… (Even best commercial ones) [BLB10, CKSW13, AGJ^{+}18, GSD19, GS19, BMN22, BBN^{+}23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

[1] See end of slides for all references with bibliographic details

WHAT CAN BE DONE ABOUT SOLVER BUGS?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Inherently can only detect presence of bugs, not absence

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  Prove that solver implementation adheres to formal specification
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- **Proof logging**
  Make solver certifying [ABM⁺11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

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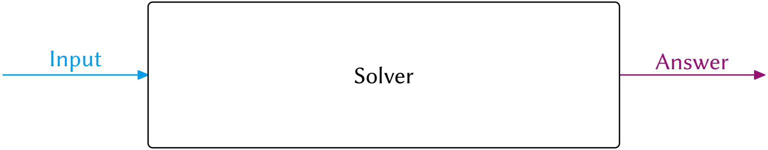
PROOF LOGGING WITH CERTIFYING SOLVERS: WORKFLOW



1. Run combinatorial solving algorithm on problem input

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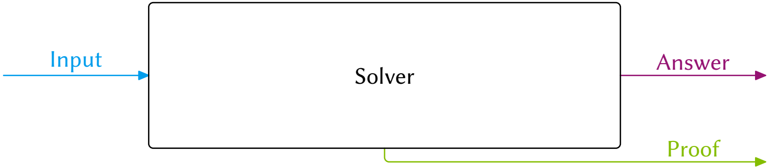

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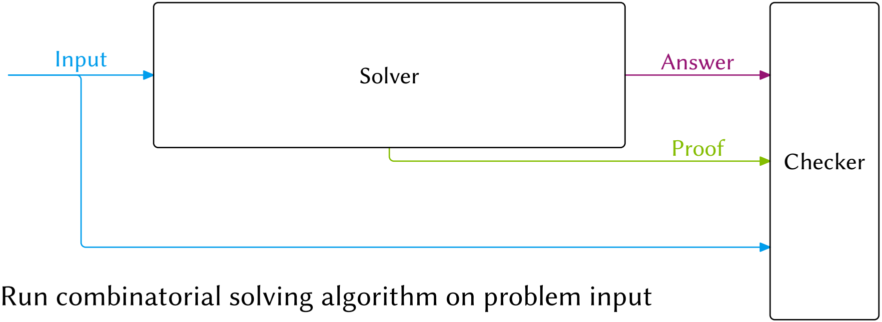

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3. Feed input + answer + proof to proof checker

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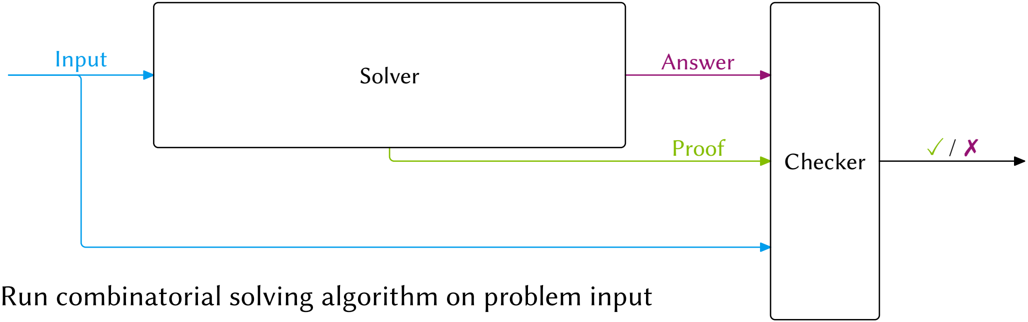
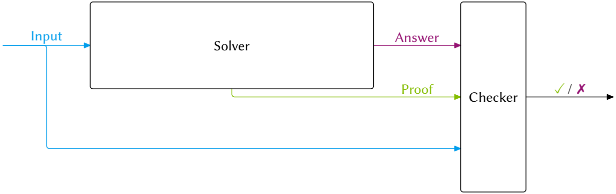


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

PROOF LOGGING DESIDERATA

Proof format for certifying solver should be



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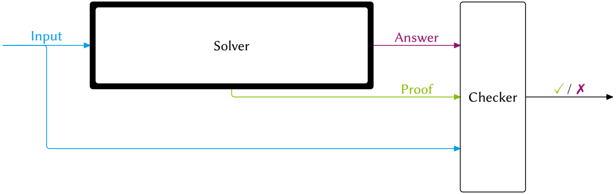

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- **very powerful:** minimal overhead for sophisticated reasoning

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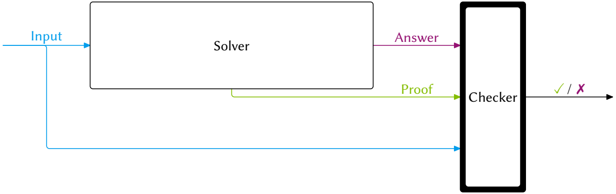

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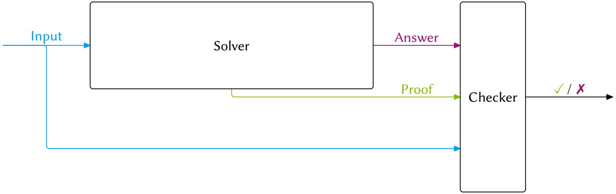

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Clear conflict expressivity vs. simplicity!

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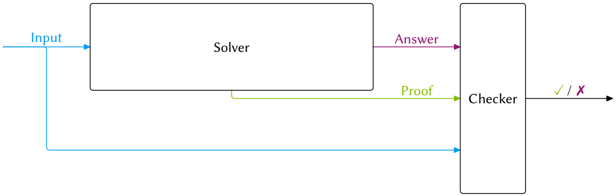

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Asking for both perhaps a little bit too good to be true?

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TAKE-AWAY MESSAGE

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Proof logging for combinatorial optimisation is possible with single, unified method!

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], …
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

THE SALES PITCH FOR PROOF LOGGING

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during development [EG21, GMM$^+$20, KM21, BBN$^+$23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

PROOF LOGGING WITH VERIPB

In extended version of this tutorial :

- SAT solving (including advanced techniques)
- SAT-based optimisation (MaxSAT)
- Subgraph algorithms
- Constraint programming
- Symmetry and dominance reasoning

in a unified way

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This seminar:

- Proof logging for SAT
- Pseudo-Boolean reasoning and cutting planes

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THE SAT PROBLEM

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

PROOFS FOR SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows "obviously" from everything we know so far
- Final clause is empty, meaning contradiction (written $\perp$)
- Means original formula must be inconsistent

WHAT IS OBVIOUS? UNIT PROPAGATION

Unit Propagation

Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ

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- $p \lor \overline{u}$ propagates $u \mapsto 0$
- $q \lor r$ propagates $r \mapsto 1$
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Proof checker should know how to unit propagate until saturation

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DAVIS-PUTMAN-LOGEMANN-LOVELAND (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

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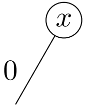

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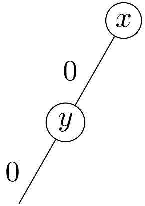

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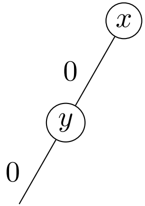

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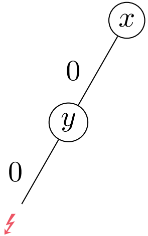

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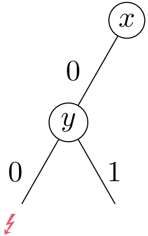

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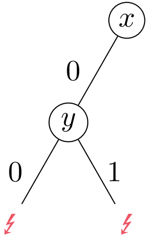

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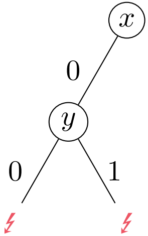
3. x



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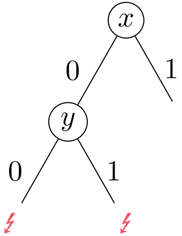

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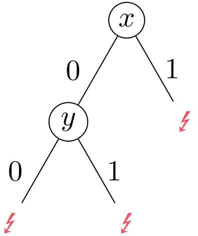
2. $x \lor \overline{y}$
3. x
4. $\overline{x}$



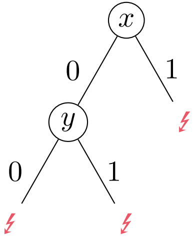
DAVIS-PUTMAN-LOGEMANN-LOVELAND (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

1. $x \vee y$
2. $x \vee \overline{y}$
3. x
4. $\overline{x}$
5. $\perp$



REVERSE UNIT PROPAGATION (RUP)

To make this a proof, need backtrack clauses to be easily verifiable

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Reverse unit propagation (RUP) clause [GN03, Van08]

C is a reverse unit propagation (RUP) clause with respect to F if

- assigning C to false
- then unit propagating on F until saturation
- leads to contradiction

If so, F clearly implies C, and this condition is easy to verify efficiently

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Fact

Backtrack clauses from DPLL solver generate a RUP proof

WHAT ABOUT CONFLICT-DRIVEN CLAUSE LEARNING (CDCL)?

Run CDCL [BS97, MS99, MMZ^{+}01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Decision

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

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$\boxed{p \stackrel{\mathrm{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathrm{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

Decision

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$\boxed{p \stackrel{\mathrm{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathrm{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

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$\boxed{q \stackrel{\mathrm{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

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$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathrm{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

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$p \stackrel{\mathrm{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathrm{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathrm{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$\boxed{p \stackrel{\mathrm{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$
decision level 1

$\boxed{q \stackrel{\mathrm{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
decision level 2

$\boxed{x \stackrel{\mathrm{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

CONFLICT ANALYSIS

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$$\left.\begin{array}{c} \boxed{p \overset{\mathrm{d}}{=} 0} \\ u \overset{p \vee \overline{u}}{=} 0 \end{array}\right\} \text{decision level 1}$$

$$\left.\begin{array}{c} \boxed{q \overset{\mathrm{d}}{=} 0} \\ r \overset{q \vee r}{=} 1 \\ w \overset{\overline{r} \vee w}{=} 1 \end{array}\right\} \text{decision level 2}$$

$$\left.\begin{array}{c} \boxed{x \overset{\mathrm{d}}{=} 0} \\ y \overset{u \vee x \vee y}{=} 1 \\ z \overset{x \vee \overline{y} \vee z}{=} 1 \\ \overset{\overline{y} \vee \overline{z}}{\perp} \end{array}\right\} \text{decision level 3}$$

CONFLICT ANALYSIS

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathrm{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

decision level 1

$q \stackrel{\mathrm{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

decision level 2

$x \stackrel{\mathrm{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

CONFLICT ANALYSIS

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathrm{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$
} decision level 1

$\boxed{q \stackrel{\mathrm{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
} decision level 2

$\boxed{x \stackrel{\mathrm{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
} decision level 3

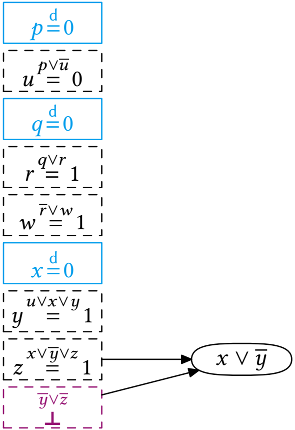
Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

CONFLICT ANALYSIS

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



Could backtrack by erasing conflict level & flipping last decision

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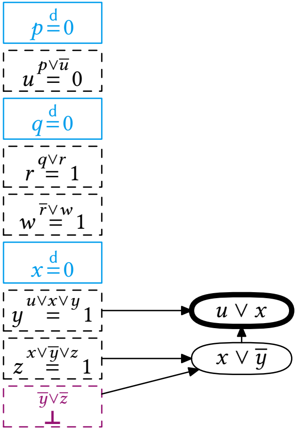
Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

CONFLICT ANALYSIS

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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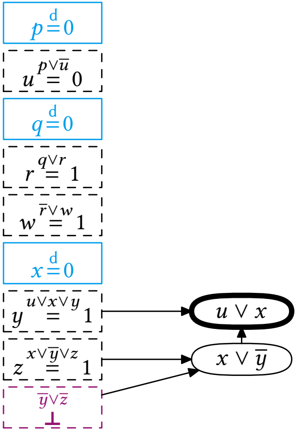
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

COMPLETE EXAMPLE OF CDCL EXECUTION

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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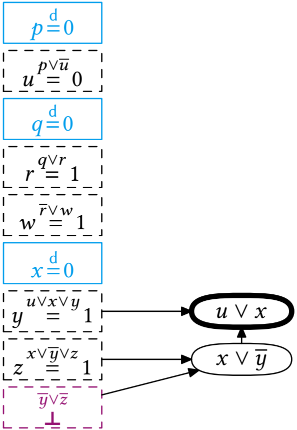

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

COMPLETE EXAMPLE OF CDCL EXECUTION

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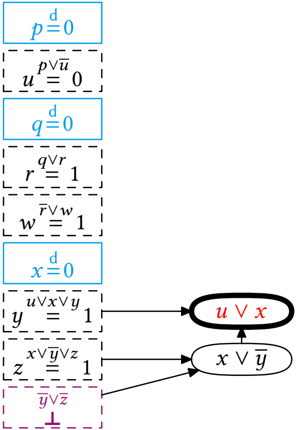
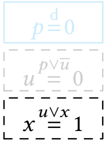
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





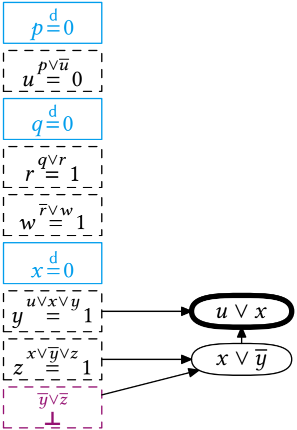
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

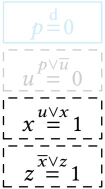
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

COMPLETE EXAMPLE OF CDCL EXECUTION

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





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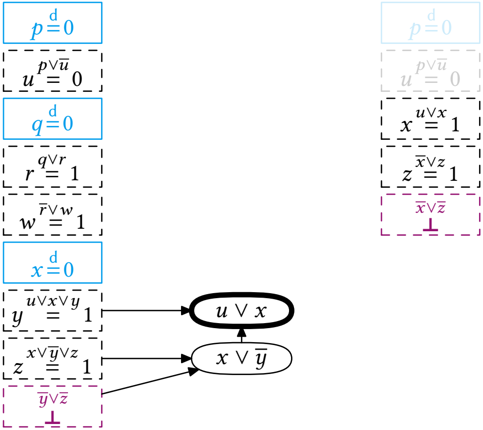
Then continue as before...

COMPLETE EXAMPLE OF CDCL EXECUTION

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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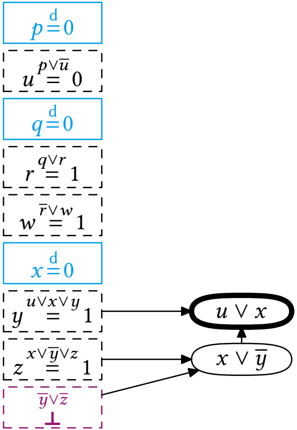

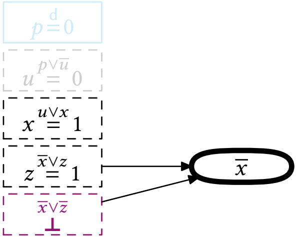

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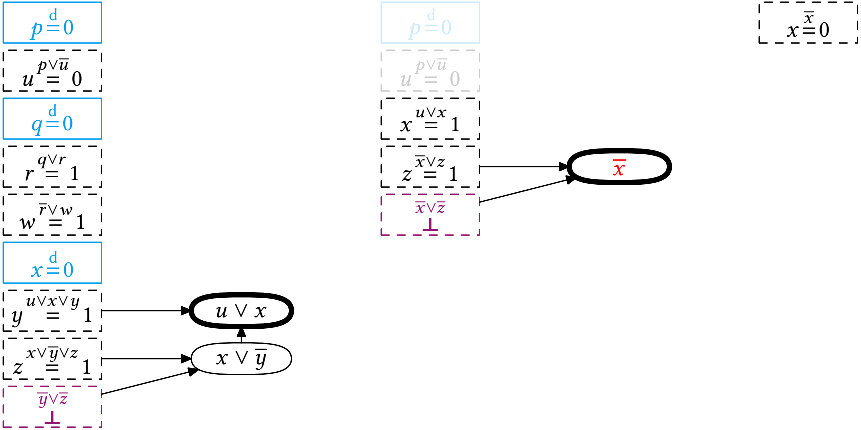

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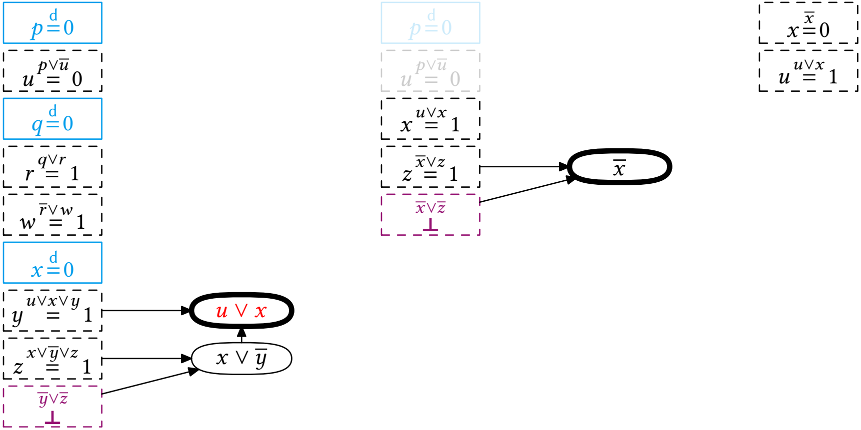

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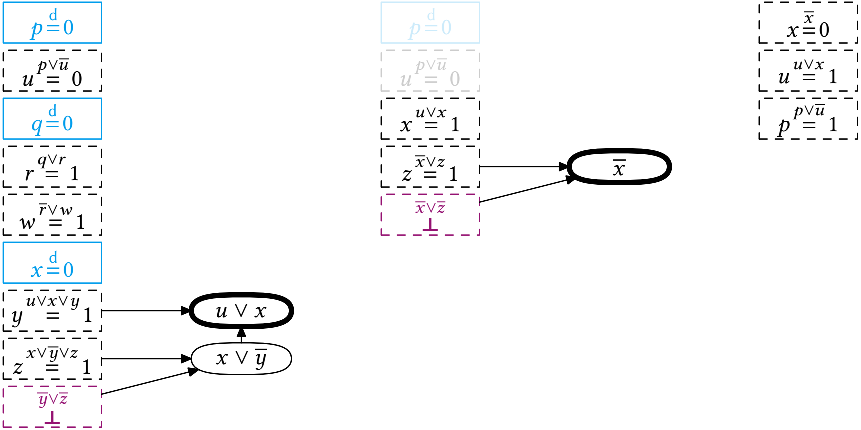

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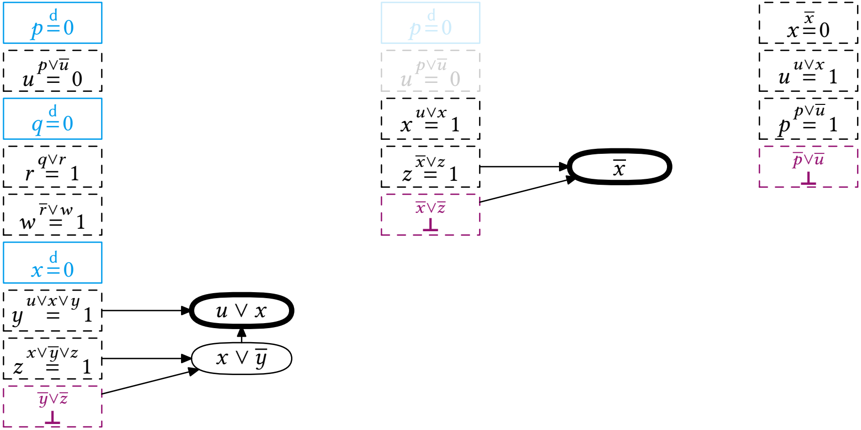

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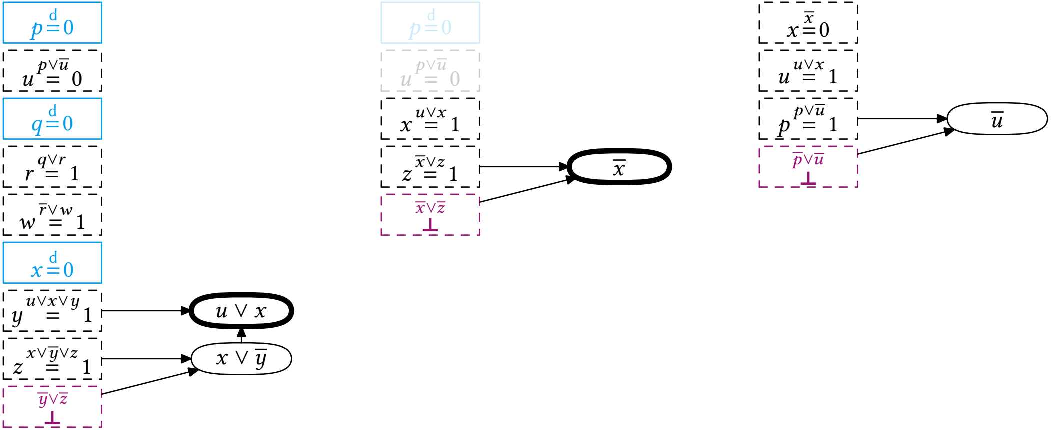

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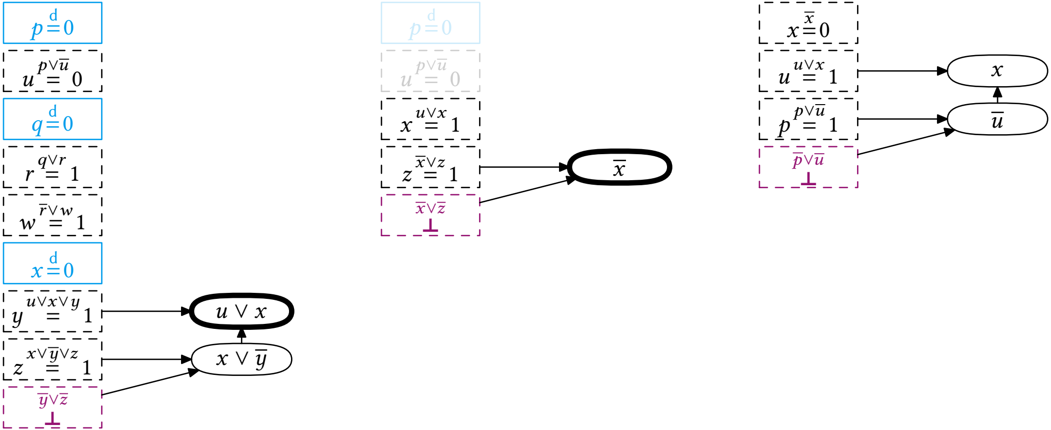

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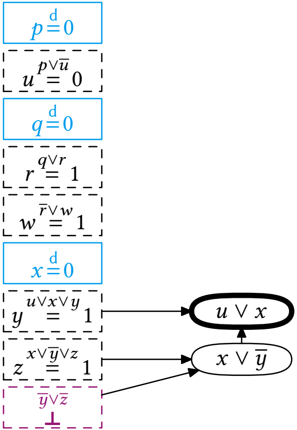

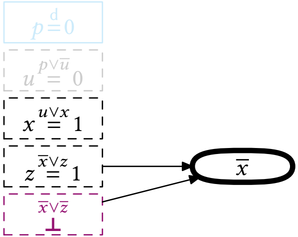

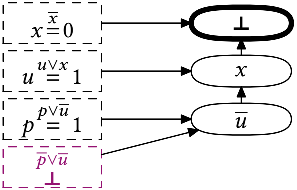

OUTLINE

1. Introduction
   1. The Success of Combinatorial Solving (and the Dirty Little Secret...)
   2. Ensuring Correctness with the Help of Proof Logging
   3. This Seminar
2. Proof Logging for SAT
   1. SAT Basics
   2. DPLL and CDCL
   3. Proof System for SAT Proof Logging
3. Pseudo-Boolean Proof Logging
   1. Pseudo-Boolean Constraints and Cutting Planes Reasoning
   2. Pseudo-Boolean Proof Logging for SAT Solving
   3. More Pseudo-Boolean Proof Logging Rules
4. Conclusions
   1. Future Work
   2. Concluding Remarks

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When run on unsatisfiable formula, CDCL generates resolution proof*

(*) Ignores pre- and inprocessing, but we will get there...

RESOLUTION PROOFS FROM CDCL EXECUTIONS

Obtain resolution proof…

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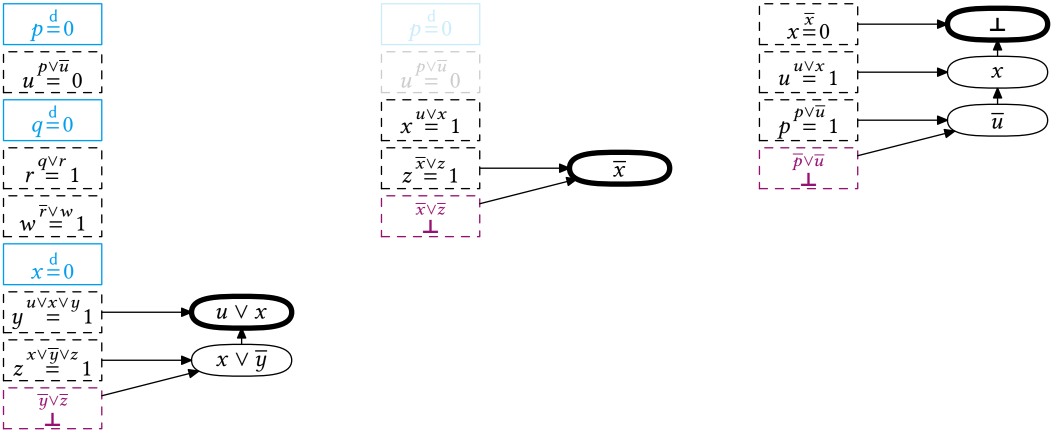

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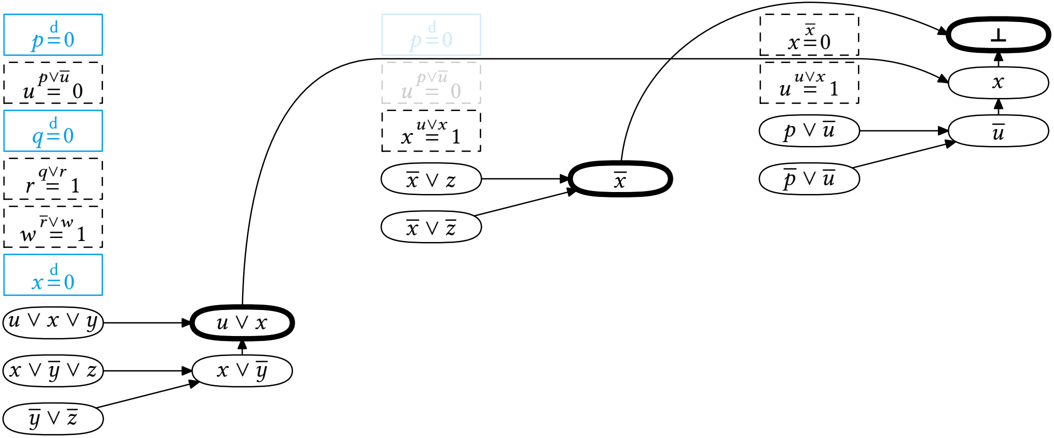

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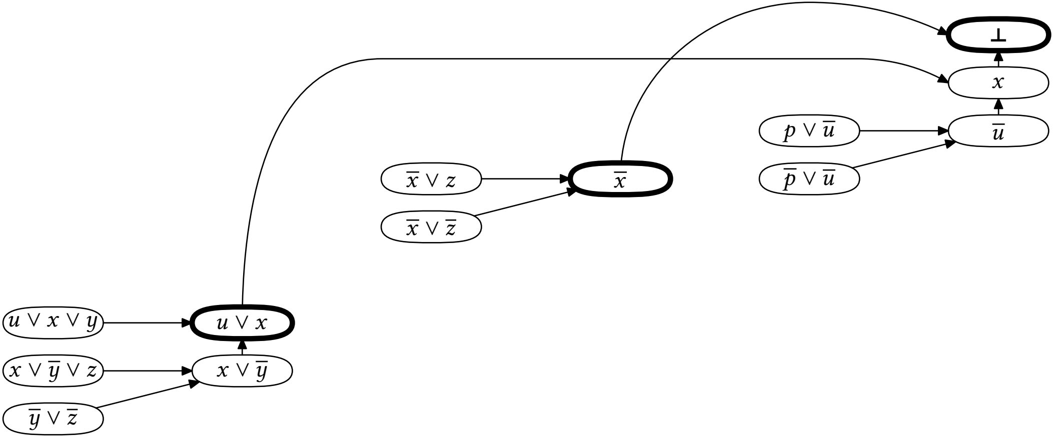

RUP PROOFS AND CDCL

But it turns out we can be lazier…

Fact

All learned clauses generated by CDCL solver are RUP clauses

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So shorter short proof of unsatisfiability for

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

is sequence of reverse unit propagation (RUP) clauses

1. $u \lor x$
2. $\overline{x}$
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MORE INGREDIENTS IN PROOF LOGGING FOR SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

EXTENSION VARIABLES, PART 1

Suppose we want a variable a encoding

$$a \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (this is fine since a doesn't appear anywhere previously)

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Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

OUTLINE

WHY AREN'T WE DONE?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can't easily reflect what algorithms for other problems do

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Surprising claim: a slight change to 0-1 integer linear inequalities does the job!

- Enables proof logging for advanced SAT techniques so far beyond reach for efficient DRAT proof logging:
  - Cardinality reasoning
  - Gaussian elimination
  - Symmetry breaking
- Supports use of SAT solvers for optimisation problems (MaxSAT)
- Can justify graph reasoning without knowing what a graph is
- Can justify constraint programming inference without knowing what an integer variable is

OUTLINE

PSEUDO-BOOLEAN CONSTRAINTS

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

SOME TYPES OF PSEUDO-BOOLEAN CONSTRAINTS

1. Clauses

$$x_1 \lor \overline{x}_2 \lor x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Input/model axioms

From the input

PSEUDO-BOOLEAN REASONING: CUTTING PLANES

Input/model axioms From the input

Literal axioms $$\frac{}{\ell_i \geq 0}$$

PSEUDO-BOOLEAN REASONING: CUTTING PLANES

Input/model axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Input/model axioms — From the input

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Multiplication for any $c \in \mathbb{N}^+$

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Division for any $c \in \mathbb{N}^+$
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

CUTTING PLANES TOY EXAMPLE

$$w + 2x + y \geq 2$$

CUTTING PLANES TOY EXAMPLE

Multiply by 2
$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

CUTTING PLANES TOY EXAMPLE

$$\text{Multiply by 2} \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

$$w + 2x + 4y + 2z \geq 5$$

CUTTING PLANES TOY EXAMPLE

$$\text{Multiply by 2}\quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

$$\text{Add}\quad \overline{\qquad 3w + 6x + 6y + 2z \geq 9 \qquad}$$

CUTTING PLANES TOY EXAMPLE

$$\text{Multiply by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5 \qquad \overline{z} \geq 0$$

$$\text{Add}\ \frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

CUTTING PLANES TOY EXAMPLE

$$\text{Add}\ \dfrac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}$$

CUTTING PLANES TOY EXAMPLE

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CUTTING PLANES TOY EXAMPLE

Multiply by 2

$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Add

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Multiply by 2

$$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$

Add

$$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Divide by 3

$$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\tfrac{1}{3}}$$

CUTTING PLANES TOY EXAMPLE

$$\text{Divide by 3}\ \dfrac{\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}}{3w + 6x + 6y \qquad \geq 7}}{w + 2x + 2y \geq 3}$$

CUTTING PLANES TOY EXAMPLE

$$
\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\text{Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\text{Multiply by 2}}{\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}\text{Divide by 3}}\text{Add}
$$

Naming constraints by integers and literal axioms by the literal involved (with $\sim$ for negation) as

$$
\begin{aligned}
\text{Constraint 1} &\doteq 2x + y + w \geq 2 \\
\text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\
\sim\! z &\doteq \overline{z} \geq 0
\end{aligned}
$$

CUTTING PLANES TOY EXAMPLE

$$\text{Divide by 3}\;\dfrac{\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Multiply by 2}\;\dfrac{w+2x+y\geq 2}{2w+4x+2y\geq 4}\qquad w+2x+4y+2z\geq 5}{3w+6x+6y+2z\geq 9}\qquad \dfrac{\overline{z}\geq 0}{2\overline{z}\geq 0}\;\text{Multiply by 2}}{3w+6x+6y\geq 7}}{w+2x+2y\geq 3}$$

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\text{Constraint 2} &\doteq 2x+4y+2z+w\geq 5\\
\sim\text{z} &\doteq \overline{z}\geq 0
\end{aligned}$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

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RESOLUTION AND CUTTING PLANES

To simulate resolution step such as

$$\frac{\overline{y} \lor \overline{z} \qquad x \lor \overline{y} \lor z}{x \lor \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add}\ \frac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{\text{Divide by 2}\ \frac{x + 2\overline{y} \geq 1}{x + \overline{y} \geq 1}}$$

RESOLUTION AND CUTTING PLANES

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \qquad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add}\ \dfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{\text{Divide by 2}\ \dfrac{x + 2\overline{y} \geq 1}{x + \overline{y} \geq 1}}$$

Given that the premises are clauses 7 and 5 in our example CNF formula, using references

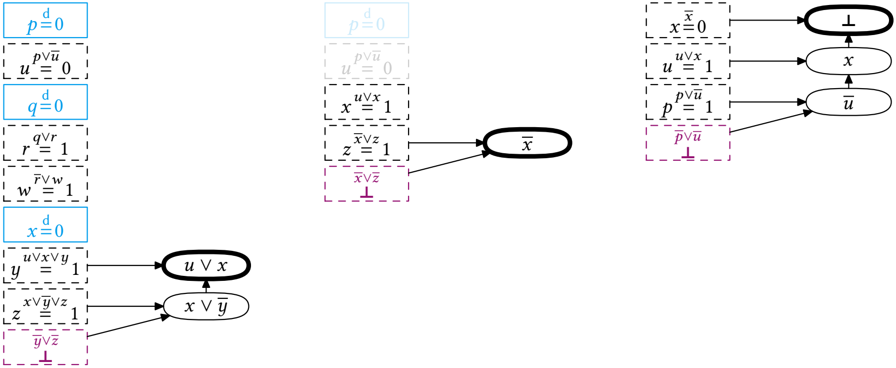
$$\text{Constraint 7} \doteq \overline{y} + \overline{z} \geq 1$$

$$\text{Constraint 5} \doteq x + \overline{y} + z \geq 1$$

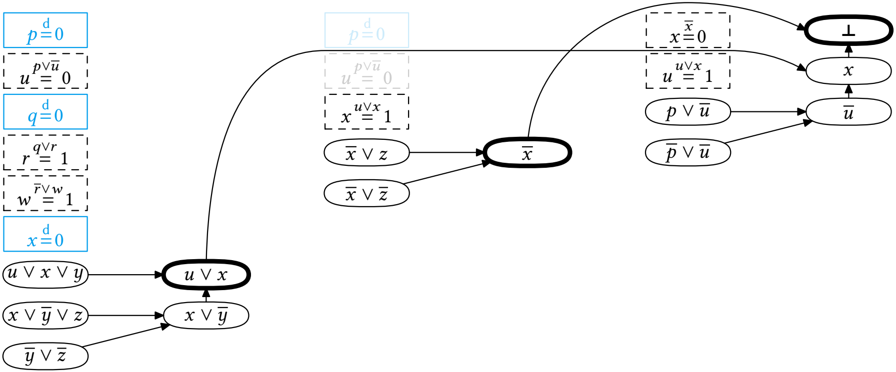
we can write this in the proof log as

```
pol 7 5 + 2 d
```

PSEUDO-BOOLEAN PROOF LOGGING FOR CDCL EXAMPLE



PSEUDO-BOOLEAN PROOF LOGGING FOR CDCL EXAMPLE



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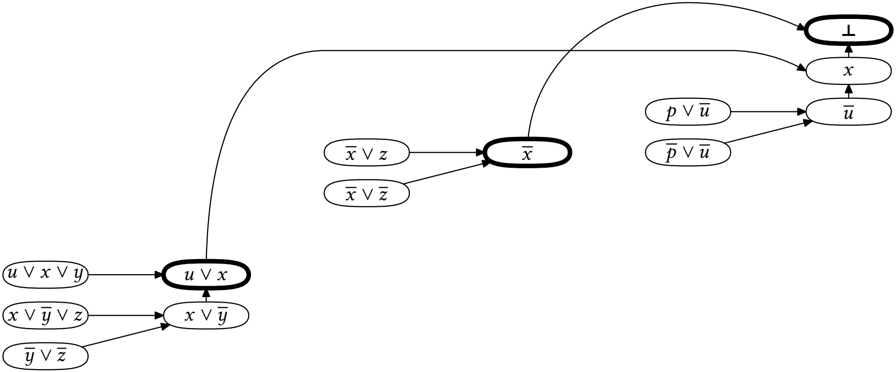
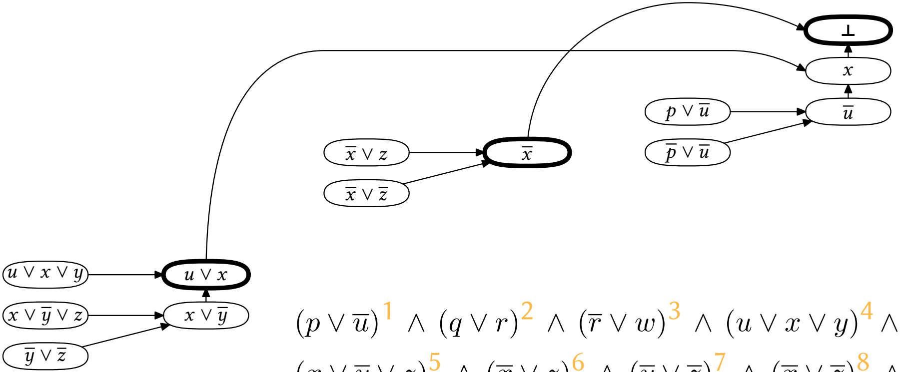


PSEUDO-BOOLEAN PROOF LOGGING FOR CDCL EXAMPLE



$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
$$(x \vee \overline{y} \vee z)^{5} \wedge (\overline{x} \vee z)^{6} \wedge (\overline{y} \vee \overline{z})^{7} \wedge (\overline{x} \vee \overline{z})^{8} \wedge (\overline{p} \vee \overline{u})^{9}$$

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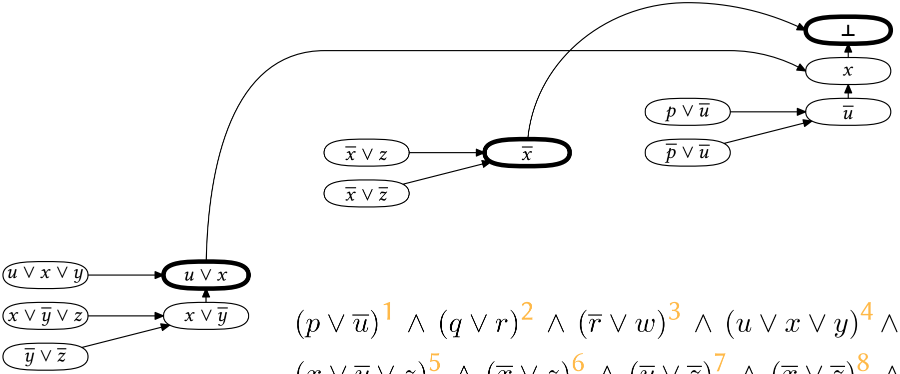

$$(p \lor \overline{u})^{1} \land (q \lor r)^{2} \land (\overline{r} \lor w)^{3} \land (u \lor x \lor y)^{4} \land$$
$$(x \lor \overline{y} \lor z)^{5} \land (\overline{x} \lor z)^{6} \land (\overline{y} \lor \overline{z})^{7} \land (\overline{x} \lor \overline{z})^{8} \land (\overline{p} \lor \overline{u})^{9}$$

```
pol 7 5 + 2 d 4 + 2 d
pol 8 6 + 2 d
pol 9 1 + 2 d 10 + 2 d 11 + 2 d
```

$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

$\rightsquigarrow$ Constraint 11 $\doteq \overline{x} \geq 1$

$\rightsquigarrow$ Constraint 12 $\doteq 0 \geq 1$ ⚡

RUP REVISITED

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint C propagates variable x if setting x to "wrong value" would make C unsatisfiable

E.g., if x_5 is false,

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

would propagate $\overline{x}_4$ (since other coefficients do not add up to 7)

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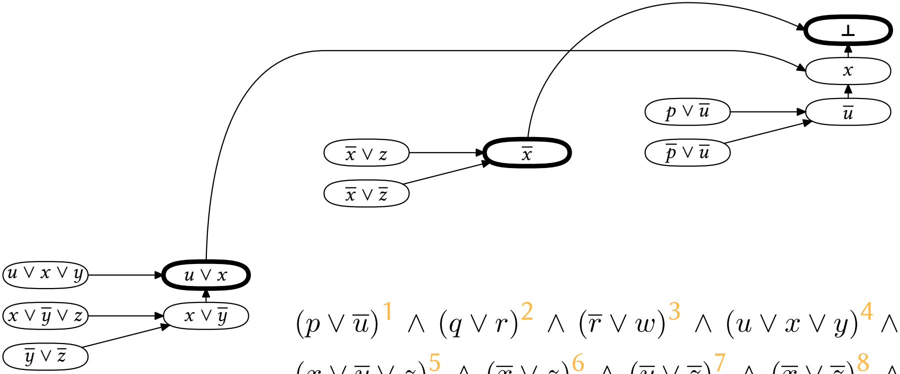
Risk for confusion:

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we're working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

PB PROOF LOGGING FOR EXAMPLE CDCL EXECUTION WITH RUP



$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
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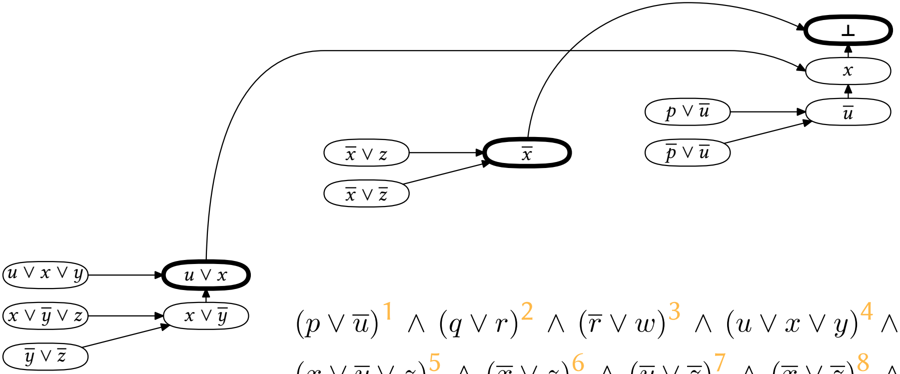

$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
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```
rup  1 u 1 x >= 1 ;
rup  1 ~x >= 1 ;
rup  >= 1 ;
```

$\rightsquigarrow$ Constraint 10 $\doteq$ $u + x \geq 1$

$\rightsquigarrow$ Constraint 11 $\doteq$ $\overline{x} \geq 1$

$\rightsquigarrow$ Constraint 12 $\doteq$ $0 \geq 1$ ϟ

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EXTENSION VARIABLES, PART 2

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system

PROOF LOGS FOR “EXTENDED CUTTING PLANES”

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

- Each constraint follows “obviously” from what is known so far
- Either implicitly, by RUP…
- Or by an explicit cutting planes derivation…
- Or as an extension variable reifying a new constraint*
- Final constraint is $0 \geq 1$

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- Final constraint is $0 \geq 1$

(*) Not actually implemented this way — details to come later…

DELETING CONSTRAINTS

In practice, important to erase constraints to save memory and time during verification

Fairly straightforward to deal with from the point of view of proof logging

So ignored in this tutorial for simplicity and clarity

ENUMERATION AND OPTIMISATION PROBLEMS

Enumeration:

- When a solution is found, can log it
- Introduces a new constraint saying “not this solution”
- So the proof semantics is “infeasible, except for all the solutions I told you about”

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For optimisation:

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise subject to the contraints in the formula
- To maximise, negate objective
- Log a solution α; get an objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: "infeasible to find better solution than best so far"

PSEUDO-BOOLEAN PROOF LOGGING — HOW AND WHY?

If problem is (special case of) 0–1 integer linear program (ILP)

- just do proof logging

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$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

THE VERIPB FORMAT AND TOOL

`https://gitlab.com/MIAOresearch/software/VeriPB`

Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- "Trust me" assertions for incremental proof logging

Documentation:

- Description of VERIPB checker [BMM+23] used in SAT 2023 competition (`https://satcompetition.github.io/2023/checkers.html`)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]
- Lots of concrete example files at `https://gitlab.com/MIAOresearch/software/VeriPB`

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Performance of pseudo-Boolean proof logging

- Trim proof while verifying (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format

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- Mixed integer linear programming *(some work on SCIP in [CGS17, EG21])*
- Satisfiability modulo theories (SMT) solving *(some work by Bjørner and others)*
- High-level modelling languages

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And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

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SUMMING UP

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- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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Thanks for your attention!

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- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

Thanks for your attention!

Interested? I'm hiring: looking for PhD students (vacancy deadline March 6th)
`https://www.kuleuven.be/personeel/jobsite/jobs/60425822`

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