Explicit back-off rates for achieving target throughputs in CSMA/CA networks

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Outline

- Model description
- Basic properties
- Questions
- Existing results
- The good
  - Line networks
  - Trees
  - Main result
- The bad
- Approximations (the ugly)

The model

Characterized by

- Undirected conflict graph $G = (V, E)$ with $n = |V|$
- Vector with back-off rates $(\nu_1, \ldots, \nu_n)$, with $\nu_i \in \mathbb{R}^+$

Evolves as

- A vertex has two states: active/inactive
- Active vertex becomes inactive after some time with mean 1 and starts a back-off period
- Back-off period of vertex i has mean length $1/\nu_i$
- Vertex becomes active at end of back-off period *if none of its neighbors* are active, otherwise new back-off period starts

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The model

Feasible states and steady state

- Let $z_i = 1$ if vertex i is active and 0 otherwise
- Set of feasible states:

$$\Omega = \{(z_1, \ldots, z_n) \in \{0,1\}^n | z_i z_j = 0 \text{ if } (i,j) \in E(G)\},$$

- Probability $\pi(\vec{z})$ to be in state $\vec{z} = (z_1, \ldots, z_n) \in \Omega$ is

$$\pi(\vec{z}) = \frac{1}{Z_n} \prod_{i=1}^{n} \nu_i^{z_i},$$

where $Z_n = \sum_{\vec{z} \in \Omega} \prod_{i=1}^{n} \nu_i^{z_i}$

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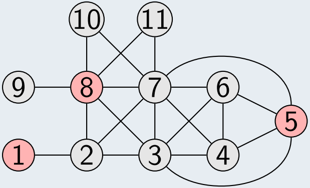
The model

Example

- Consider conflict graph G (red is active)



- Probability to be in the above state is

$$\pi(\vec{z}) = \frac{1}{Z_n} \nu_1 \nu_5 \nu_8,$$

$\Rightarrow \pi(\vec{z})$ is proportional to product of rates of active vertices

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The model

Throughput vector

- Throughput of vertex i is fraction of time that vertex i is active:
$$\theta_i = \sum_{\vec{z} \in \Omega, z_i = 1} \pi(\vec{z}).$$
- Set of achievable throughput vectors (Jiang and Walrand, 2010)
$$\Gamma = \left\{ \sum_{\vec{z} \in \Omega} \xi(\vec{z}) \vec{z} \,\middle|\, \sum_{\vec{z} \in \Omega} \xi(\vec{z}) = 1, \xi(\vec{z}) > 0 \text{ for } \vec{z} \in \Omega \right\}.$$
- For any $\vec{\theta} \in \Gamma$ there exists a unique vector of back-off rates $\vec{\nu}(\vec{\theta})$ that achieves $\vec{\theta}$ (van de Ven, Janssens, van Leeuwaarden and Borst, 2011)

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The model

Throughput vector

- Let $\mathcal{N}_i = \{j | (i,j) \in E(G)\}$ be neighbors of i
- Let Z_W with $W \subset V(G)$ be the normalizing constant of the network induced by W
- Throughput of vertex i can be expressed as

$$\theta_i = \frac{\nu_i Z_{V(G) \setminus (\mathcal{N}_i \cup i)}}{Z_n}$$

for example:

$$\theta_8 = \nu_8 Z_{\{1,4,5,6\}} / Z_n = \nu_8 (1 + \nu_1)(1 + \nu_4 + \nu_5 + \nu_6) / Z_n$$

- Proof idea: find explicit expressions for Z_n and $Z_{V(G) \setminus (\mathcal{N}_i \cup i)}$

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The model

Questions

- For which graphs G can we find an explicit expression for the unique vector of back-off rates $\vec{\nu}(\vec{\theta})$?
- Can we compute these rates in a distributed manner?
- How about general conflict graphs?

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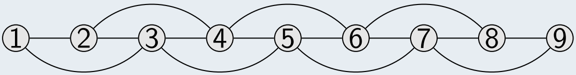
Existing results

Fairness in line networks
(van de Ven, Janssens and van Leeuwaarden, 2009)

- Line networks with interference range $\beta \in \{1, 2, \ldots, n-1\}$.
- Example: $n = 9$, $\beta = 2$



- When $\theta_1 = \ldots = \theta_n = \gamma < 1/(\beta+1)$,

$$\nu_i(\vec{\theta}) = \gamma \frac{(1-\gamma\beta)^{h_i - 1}}{(1-\gamma(\beta+1))^{h_i}},$$

  with $h_i = \min(i+\beta, n) - \max(i, \beta+1) + 1$.
- For $n = 9, \beta = 2$: $(h_1, \ldots, h_9) = (1, 2, 3, 3, 3, 3, 3, 2, 1)$

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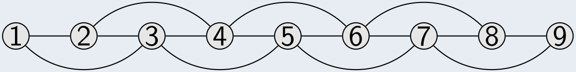
The good

Line networks: $\vec{\theta} \in \Gamma$

- Example: $n = 9$, $\beta = 2$



- Back-off rates for node $1, 2$ and $i = 3, \ldots, 7$ is

$$\nu_1(\vec{\theta}) = \frac{\theta_1}{1 - \theta_1 - \theta_2 - \theta_3}$$

$$\nu_2(\vec{\theta}) = \frac{\theta_2(1 - \theta_2 - \theta_3)}{(1 - \theta_1 - \theta_2 - \theta_3)(1 - \theta_2 - \theta_3 - \theta_4)}$$

$$\nu_i(\vec{\theta}) = \frac{\theta_i(1 - \theta_{i-1} - \theta_i)(1 - \theta_i - \theta_{i+1})}{(1 - \theta_{i-2} - \theta_{i-1} - \theta_i)(1 - \theta_{i-1} - \theta_i - \theta_{i+1})(1 - \theta_i - \theta_{i+1} - \theta_{i+2})}$$

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The good

Line networks: $\vec{\theta} \in \Gamma$

- Line networks with interference range $\beta \in \{1, 2, \ldots, n-1\}$.
- For $\vec{\theta} \in \Gamma$, we have

$$\nu_i(\vec{\theta}) = \theta_i \frac{\prod_{j=\max(i,\beta+1)}^{\min(i+\beta,n)-1} (1 - \theta_{j-\beta+1} - \ldots - \theta_j)}{\prod_{j=\max(i,\beta+1)}^{\min(i+\beta,n)} (1 - \theta_{j-\beta} - \ldots - \theta_j)},$$

- $\vec{\theta} \in \Gamma$ if and only if

$$T = \max_{i=1}^{n-\beta} (\theta_i + \ldots + \theta_{i+\beta}) < 1$$

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The good[1]

Tree networks (i.e., acyclic conflict graph G)

- For $\vec{\theta} \in \Gamma$, we have

$$\nu_i(\vec{\theta}) = \frac{\theta_i(1-\theta_i)^{|\mathcal{N}_i|-1}}{\prod_{j\in\mathcal{N}_i}(1-\theta_i-\theta_j)},$$

- $\vec{\theta} \in \Gamma$ if and only if

$$T = \max_{(k,j)\in E}(\theta_k+\theta_j) < 1$$

- Back-off rate of vertex i depends only on θ_i and the target throughputs of neighbors in G

[1] This formula was presented earlier by Yun, Shin and Yi (2015, IEEE Trans. Inf. Theory) as the Bethe approximation

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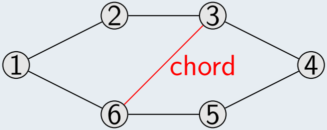
Chordal graphs

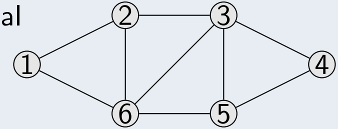
Definition

- A graph G is chordal if and only if all cycles of length > 3 have a *chord*
- A *chord* of a cycle C is an edge joining two nonconsecutive vertices of C

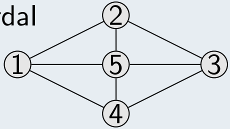


Examples

chordal



not chordal



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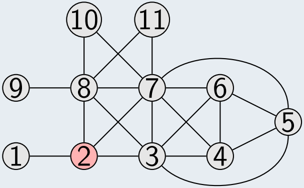
Chordal graphs

Perfect elimination order (peo)

- A peo is an ordering of $V(G)$ such that for $v \in V(G)$ we have *v and the neighbors of v that appear after v in the order form a clique*
- Example:



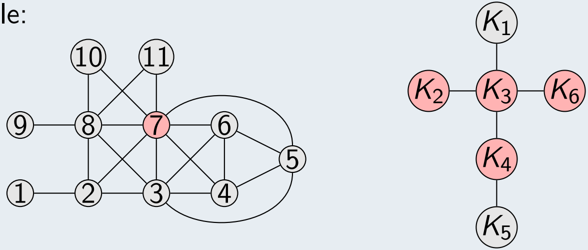
peo = 1, 9, 10, 11, 8, 2, 3, 4, 5, 7, 6

- G is chordal if and only if it has a peo
- peo can be found in $O(|V(G)| + |E(G)|)$ time

Chordal graphs

Clique tree

- A clique tree $T = (\mathcal{K}_G, \mathcal{E})$ for G is a tree in which
  - $\mathcal{K}_G$ corresponds to the maximal cliques of G
  - $\mathcal{E}$ is such that the subgraph of T induced by the maximal cliques that contain v is a subtree of T for any $v \in V$
- Example:



$K_1 = \{1, 2\}, K_2 = \{3, 4, 5, 6, 7\}, K_3 = \{2, 3, 7, 8\}, K_4 = \{7, 8, 10\}, K_5 = \{8, 9\}, K_6 = \{7, 8, 11\}$

- G is chordal if and only if it has at least one clique tree

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Main result: chordal conflict graphs

Clique tree representation

- Let G be chordal and $T = (\mathcal{K}_G, \mathcal{E})$ is a clique tree of G:

$$\nu_i(\vec{\theta}) = \theta_i \frac{\prod_{(K,K')\in\mathcal{E}, i\in K\cap K'} \left(1 - \sum_{s\in K\cap K'} \theta_s\right)}{\prod_{K\in\mathcal{K}_G, i\in K} \left(1 - \sum_{s\in K} \theta_s\right)},$$

- $\vec{\theta} \in \Gamma$ if and only if $T = \max_{j\in\mathcal{K}_G} \sum_{s\in K_j} \theta_s < 1$
- Back-off rate of vertex i depends only on θ_i and the target throughputs of neighbors in G

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Algorithm for chordal conflict graphs

Peo algorithm

Input: A chordal conflict graph G

Output: Back-off rates $\nu_1(\vec{\theta}), \ldots, \nu_n(\vec{\theta})$

Determine a perfect elimination ordering of G

for $i = 1$ *to* n **do**

  Let $\alpha(i)$ be the node in position i in this order;

end

for $i = 1$ *to* n **do**

  Let $\mathcal{M}_{\alpha(i)} = \mathcal{N}_{\alpha(i)} \cap \{\alpha(i+1), \ldots, \alpha(n)\}$;

end

$\nu_{\alpha(n)}(\vec{\theta}) = \theta_{\alpha(n)}/(1 - \theta_{\alpha(n)})$;

for $i = n - 1$ **down to** 1 **do**

  $\nu_{\alpha(i)}(\vec{\theta}) = \theta_{\alpha(i)}/(1 - \theta_{\alpha(i)} - \sum_{s \in \mathcal{M}_{\alpha(i)}} \theta_s)$;

  for $j \in \mathcal{M}_{\alpha(i)}$ **do**

    $\nu_j(\vec{\theta}) = \nu_j(\vec{\theta}) \frac{1 - \sum_{s \in \mathcal{M}_{\alpha(i)}} \theta_s}{1 - \theta_{\alpha(i)} - \sum_{s \in \mathcal{M}_{\alpha(i)}} \theta_s}$;

  end

end

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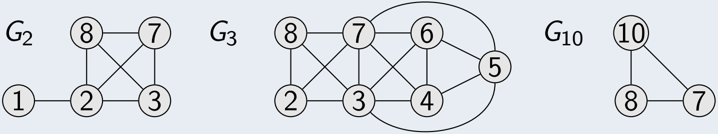
Distributed algorithm for chordal conflict graphs

Local peo representation

- Each vertex i sends θ_i and the list $\mathcal{N}_i \cup \{i\}$ to $j \in \mathcal{N}_i$
- To set rate $\nu_i(\vec{\theta})$
  - Construct the subgraph G_i of G induced by vertices $\mathcal{N}_i \cup \{i\}$
  - Run peo algorithm on G_i to determine $\nu_i(\vec{\theta})$

Example



$$\Rightarrow \nu_2(\vec{\theta}) = \frac{\theta_2(1-\theta_2)}{(1-\theta_2-\theta_3-\theta_7-\theta_8)(1-\theta_1-\theta_2)}, \ \nu_{10}(\vec{\theta}) = \frac{\theta_{10}}{1-\theta_7-\theta_8-\theta_{10}}$$

$$\Rightarrow \nu_3(\vec{\theta}) = \frac{\theta_3(1-\theta_3-\theta_7)}{(1-\theta_2-\theta_3-\theta_7-\theta_8)(1-\theta_3-\theta_4-\theta_5-\theta_6-\theta_7)}$$

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Beyond chordal conflict graphs: the bad

Ring networks: back-off rate depends on all θ's

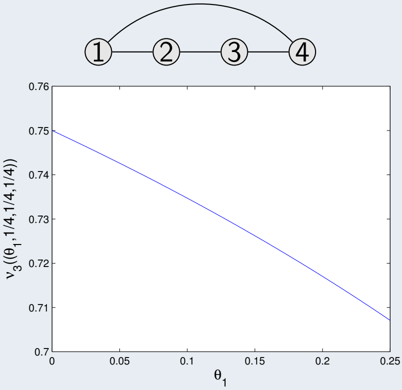

Figure: Back-off rate of node 3 as a function of θ_1 in a ring network with $n = 4$ nodes, where $\theta_2 = \theta_3 = \theta_4 = 1/4$.

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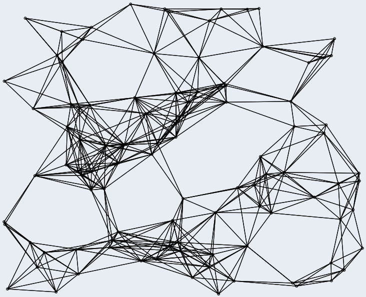
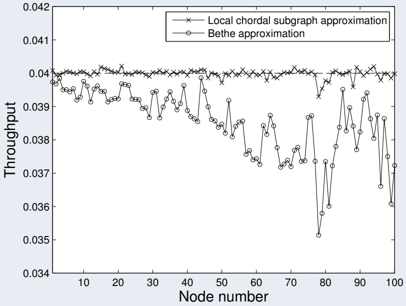
Distributed approximation for general conflict graphs

Local chordal subgraph approximation

- Find a maximal chordal subgraph $\tilde{G}_i$ of G_i via MAXCHORD algorithm[a] with i as initial vertex
- Set $\nu_i(\vec{\theta})$ by running peo algorithm on this subgraph $\tilde{G}_i$





[a] Dearing, Shier and Warner, 1988

Further reading

this talk

- http://arxiv.org/abs/1602.08290

my work

- http://win.ua.ac.be/~vanhoudt/

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