Transducers and Their Decision Problems

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No free lunch seminar @ UAntwerp

String-to-String Transductions

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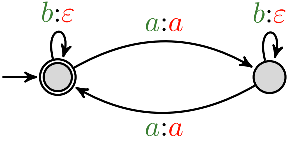
Mirror the input word	$antwerp \mapsto prewtna$
Copy the input word	$yes \mapsto yesyes$

Transductions are defined by Transducers

Transducers: Automata with output

f_{del} :



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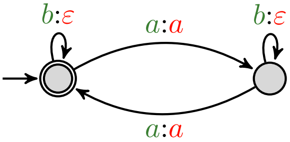

$$aabaa \quad \mapsto \quad aaaa$$

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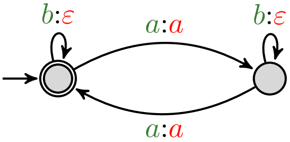

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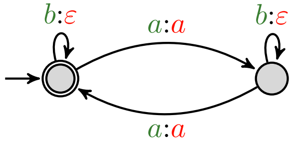
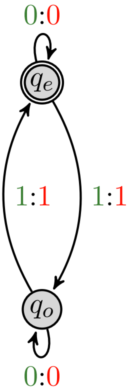


$$\begin{array}{rcl} aabaa & \mapsto & aaaa \\ aaba & \mapsto & \text{undefined} \\ dom(f_{del}) & = & \text{`even number of } a\text{'} \end{array}$$

Parity bit

$01101 \mapsto \mathbf{1}01101, 01111 \mapsto \mathbf{0}01111$



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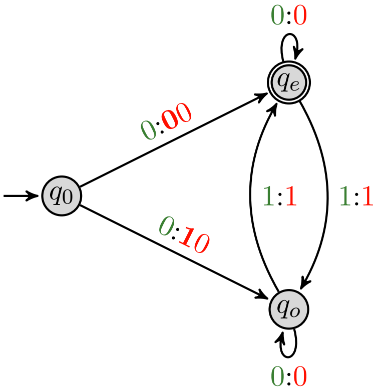

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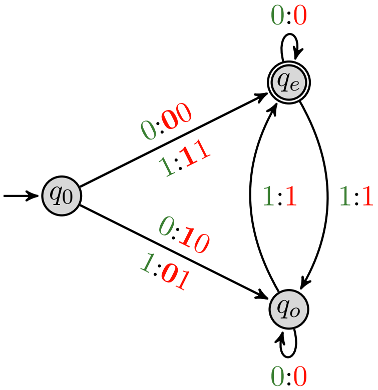
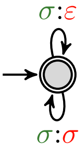


Non-determinism and relations

In general, transducers define binary *relations* in $\Sigma^* \times \Sigma^*$



defines $\{(u, v) \mid v \text{ is a subword of } u\}$

Formal Definition

Definition

A (finite state) *transducer* is a tuple $T = (\Sigma, Q, I, F, \Delta)$ where:

- Σ is a finite alphabet
- Q is a finite set of states
- $I \subseteq Q$ are the initial states and $F \subseteq Q$ are the final states
- $\Delta \subseteq Q \times \Sigma \times \Sigma^* \times Q$ is the transition relation.

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Semantics

A *run* is a sequence of transitions

$$r = q_0 \xrightarrow{\sigma_1 : v_1} q_1 \ldots q_{n-1} \xrightarrow{\sigma_n : v_n} q_n \qquad \sigma_i \in \Sigma$$

Its input is $in(r) = \sigma_1 \ldots \sigma_n$ and its output $out(r) = v_1 \ldots v_n$.
The (rational) *relation* defined by T is:

$$[\![T]\!] = \{(in(r), out(r)) \mid r \text{ is an accepting run}\}$$

Application: Natural Language Processing

Table of nouns and their plural forms

Noun singular	Noun plural
cat	cats
dog	dogs
cow	cows
fox	foxes
bus	buses
quiz	quizzes
goose	geese
spy	spies
city	cities
...	...

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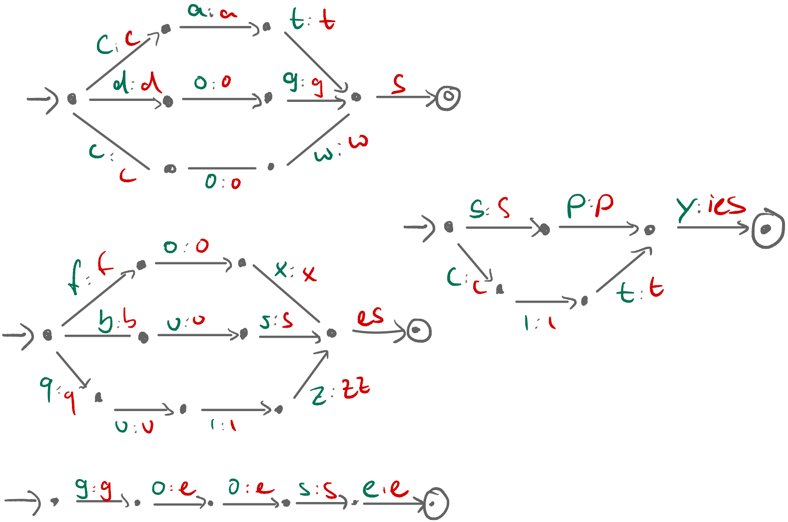
- Inefficient representation

Lexicon + spelling rules: Transducer for plural forms



Properties of Transducers

Closure Properties

- Union

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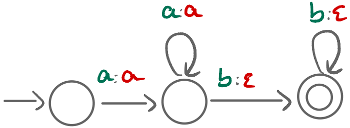
- Union ✓
- Intersection

Closure Properties: Intersection

1. show that $\{(a^n b^m, a^n) \mid n, m > 0\}$ is rational.



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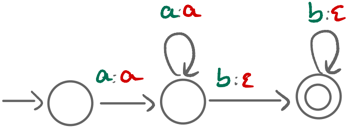

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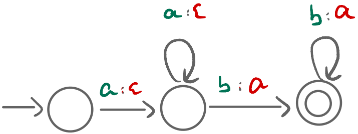

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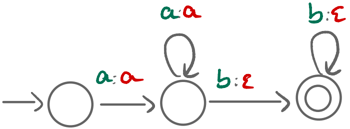

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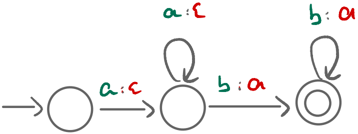

3. are rational relations closed under intersection ? why ?

$$\cap := \{(a^n b^m, a^n) \mid n = m > 0\}$$

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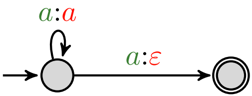

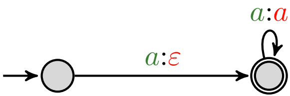

- Consider r_1, r_2 two runs on a^3. We have $(in(r_1), out(r_1)) = (in(r_2), out(r_2))$ but different in-out words:
$$(a, a)(a, a)(a, \varepsilon) \neq (a, \varepsilon)(a, a)(a, a)$$

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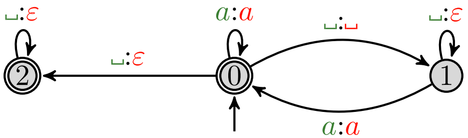
$$(a, a)(a, a)(a, \varepsilon) \neq (a, \varepsilon)(a, a)(a, a)$$
- Transducers are *asynchronous*
- Make most transducer problems conceptually difficult (and even computationally).

Determinizability: Example



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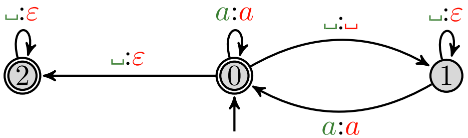

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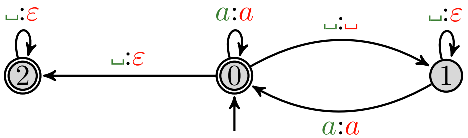

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Is non-determinism needed ?

Determinizability: Example

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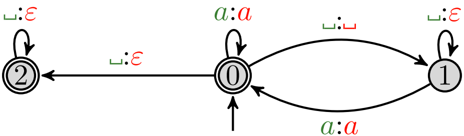

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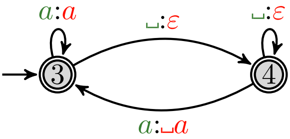

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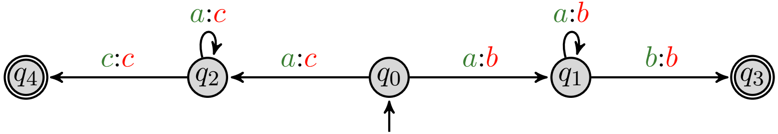
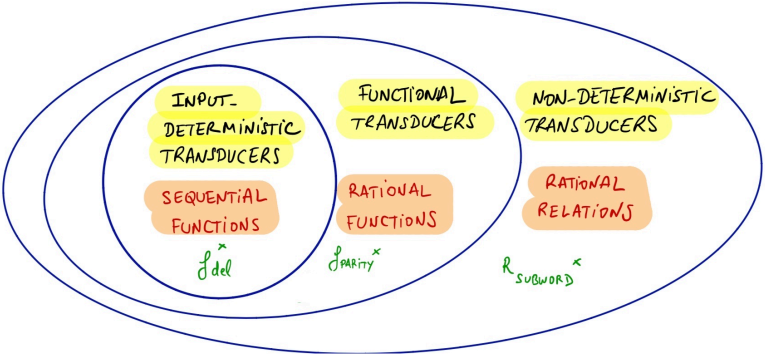


Semantics

$$[\![T]\!] : \begin{cases} a^n b \mapsto b^{n+1} \\ a^n c \mapsto c^{n+1} \end{cases}$$

functional but not determinizable

Different classes of transductions



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- Equivalence reduces to functionality:
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General case

- Undecidable (Griffith 68),
- even if one alphabet is unary (Ibarra 78)

Summary – Transducers

Expressiveness:

input-deterministic	functional	non-deterministic
f_{del}	$<$ f_{parity}	$<$ $R_{subword}$

PTime PTime

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input-deterministic	functional	non-deterministic
f_{del} $<$	f_{parity} $<$	$R_{subword}$

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Equivalence: ($dom(T_1) = dom(T_2)$ is known)

input-deterministic	functional	non-deterministic
PTime	PTime	undec